2.2 – Separable Equations

DEFINITION 1.1.1 Differential Equation

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE).**

DEFINITION 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

A DE of the form $\frac{dy}{dx} = g(x)\,h(y)$ is said to be separable.

To solve, we separate variables and integrate:

Note: $y=0$ is a singular solution.

6. $\frac{dy}{dx} + 2xy^2 = 0$ $\Rightarrow$ $\frac{dy}{dx} = -2xy^2$

$$\int y^{-2}\,dy = \int -2x\,dx$$

$$-\frac{1}{y} + \cancel{C} = -x^2 + \cancel{C}$$

$$-\frac{1}{y} = -x^2 + C_1$$

when $x=2,\ y=\frac{1}{2}$

$$-2 = -4 + C_1$$

$$C_1 = 2$$

Let $C = -C_1$

$$\frac{1}{y} = x^2 + C$$

This is a one-parameter family of solutions.

$$\boxed{y = \frac{1}{x^2+C}}$$

initial condition

What if we know that $y(2) = \frac{1}{2}$?

Then $y = \frac{1}{2}$ when $x = 2$.

$$\frac{1}{2} = \frac{1}{4+C} \Rightarrow C = -2$$

$$y = \frac{1}{x^2-2}$$

The problem $\frac{dy}{dx} + 2xy^2$, $y(2) = \frac{1}{2}$ is called an initial-value problem.

8. $e^x y \dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

$$e^x y \frac{dy}{dx} = e^{-y}(1 + e^{-2x})$$

$$y e^y dy = (e^{-x} + e^{-3x})\, dx$$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$\frac{1}{2}\int 2e^{2x}\,dx$ $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

$$\boxed{ye^y - e^y = -e^{-x} - \tfrac{1}{3}e^{-3x} + C}$$

This is an <u>implicit solution</u>.

The previous solution, $y = \frac{1}{x^2+C}$ is an <u>explicit solution</u>.

In Problems 31–34 find an explicit solution of the given initial-value problem. Determine the exact interval I of definition by analytical methods. Use a graphing utility to plot the graph of the solution.

31. $\frac{dy}{dx} = \frac{2x+1}{2y}$, $\quad y(-2) = -1$

32. $(2y-2)\frac{dy}{dx} = 3x^2 + 4x + 2$, $\quad y(1) = -2$

1^{st}, find an implicit solution.

$$(2y-2)\,dy = (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

When $x=1, y=-2$ $\quad 4+4 = 1+2+2+C \Rightarrow C = 3$

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 3 + 1$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$y-1 = \pm\sqrt{x^3+2x^2+2x+4}$$

$$y = 1 \pm \sqrt{x^3+2x^2+2x+4}$$

since $y <$ when $x=1$, we pick minus

$$\boxed{y = 1 - \sqrt{x^3+2x^2+2x+4}}$$

this is defined if $x^3+2x^2+2x+4 \geq 0$

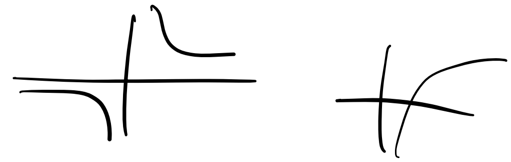
$$x^2(x+2)+2(x+2) \geq 0$$

$$(x^2+2)(x+2) \geq 0$$

This is satisfied if $x \geq -2$.

$$\frac{dy}{dx} = -\frac{\text{stuff}'}{2\sqrt{\text{stuff}}} \quad \leftarrow x > -2$$

Interval of solution : $(-2, \infty)$



$$\int \frac{1}{x}\,dx = \ln|x| + C$$

16. $\frac{dQ}{dt} = k(Q - 70)$

$$\int \frac{dQ}{Q-70} = \int k\,dt$$

$$\ln|Q-70| = (kt + C_1)$$

$$e \qquad\qquad e$$

$$|\pm(Q-70)| = e^{kt}e^{C_1}$$

Let $C_2 = e^{C_1}$

$$Q - 70 = C e^{kt}$$

let $C = \pm C_2$

$$\boxed{Q = 70 + Ce^{kt}}$$

Aside

$|4| = 4$

$|-4| = 4$

$|\pm 4| = 4$

$|\pm x| = 4$

$x = \pm 4$

30. $\frac{dy}{dx} = y^2 \sin x^2, \quad y(-2) = \frac{1}{3}$

$$y^{-2}\,dy = \sin x^2\,dx$$

t is a dummy variable

$$\int_{-2}^{x} y^{-2}\,dy = \int_{-2}^{x} \sin t^2\,dt$$

$$-\frac{1}{y(t)}\Bigg|_{-2}^{x} = \int_{-2}^{x} \sin t^2 \, dt$$

$$-\frac{1}{y(x)} + \frac{1}{y(-2)}$$

$$-\frac{1}{y(x)} + 3 = \int_{-2}^{x} \sin t^2 \, dt$$

$$y(x) = \left(3 - \int_{-2}^{x} \sin t^2 \, dt\right)^{-1}$$